MIXING OF MUTUALLY SOLUBLE LIQUIDS IN TURBULENT FLOW IN A PIPE

V. I. Maron

UDC 532.507

The problem concerned with mixing of mutually soluble liquids in turbulent flow in a pipe [1-11] is considered. To describe the distribution of concentration in the region of mixture, taken average across the section of the pipe, we use a model based on a one-dimensional model of the type of heat-conduction equation with an effective coefficient which, as tests show, is different from the coefficients of molecular and turbulent transfer. The dimensionless value of this coefficient depends on a number of parameters, such as the Reynolds number calculated for one of the liquids, roughness, ratio of the densities and viscosities of the liquids, as well as on the concentration, gradients of concentration, etc. These relationships can be established either by means of tests or on the basis of theoretical consideration of the mixing phenomenon. In this paper we theoretically derive a dispersion model with an effective diffusion coefficient which depends on Reynolds and Schmidt numbers, as well as on roughness.

1. We consider the combined turbulent motion of two mutually soluble liquids in a circular pipe, when one of them moves after the other or when a portion of one liquid is propagating in the basic stream of the other liquid. During such a motion, because of convective transfer of matter with a velocity varying across the diameter of the pipe, and turbulent diffusion, mixing of the liquids and formation of a mixture region separating the homogeneous components take place. In a fully developed turbulent flow, stratification of the liquids in the gravitation field will be insignificant because of the intensity of mixing. Therefore the flow is taken as axisymmetric.

The axisymmetric concentration distribution in turbulent flow in a circular pipe is described by a diffusion equation of the following form:

$$\begin{gathered}\frac{\partial c}{\partial t}+U\Phi(r)\frac{\partial c}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left(rD\frac{\partial c}{\partial r}\right)\\ c=c(t,x,r),\quad t>0,\quad 0<r<a,\quad -\infty<b<x<d<+\infty\end{gathered}\tag{1.1}$$

Here c is the averaged concentration, U is the mean velocity of flow, $U\Phi(r)$ is the profile of the averaged velocity, $D(r)$ is the transfer coefficient, a is the radius of the pipe, and x and r are the spatial variables. The x axis of the cylindrical coordinate system coincides with axis of the pipe and is directed towards the motion of the stream.

A term taking into account the longitudinal diffusion is absent from the right side of this equation. Taylor showed that the longitudinal diffusion need not be considered, when determining the concentration profile in the first approximation. In the following, after a one-dimensional model has been constructed, it will not be difficult to take into account the correction to the virtual coefficient from the axial diffusion. This correction turns out to be very small [2].

The initial and boundary conditions of the problem, when the flow of the material on the inner surface of the pipe is zero, have the following form:

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 96-102, September-October, 1971. Original article submitted April 14, 1971.

$$c(0, x, r) = c_0(x), \quad \left.\frac{\partial c}{\partial r}\right|_{r=0} = \left.\frac{\partial c}{\partial r}\right|_{r=a} = 0 \tag{1.2}$$

To these conditions we must add the corresponding limiting conditions for $x = b$ and $x = d$.

We introduce the dimensionless variables

$$\tau = D_0 / a^2 t, \qquad \xi = x / a, \qquad \eta = r / a \tag{1.3}$$

By means of these variables the diffusion equation and the limiting conditions of the problem are written in the form

$$\begin{gathered}\frac{\partial c}{\partial \tau} + Y \frac{\partial c}{\partial \xi} + Y\,[\Phi(\eta) - 1]\,\frac{\partial c}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta D_* \frac{\partial c}{\partial \eta}\right)\\ c = c(\tau, \xi, \eta), \quad Y = aU / D_0, \quad D_* = D / D_0\\ c\left.\right|_{\tau=0} = c_0(\xi), \quad \left.\frac{\partial c}{\partial \eta}\right|_{\eta=0} = \left.\frac{\partial c}{\partial \eta}\right|_{\eta=1} = 0\end{gathered} \tag{1.4}$$

Here D_0 is a typical value of the transfer coefficient.

The terms on the left and right sides of Eq. (1.4) are multiplied by 2η and are integrated with respect to η between the limits 0 and 1. We have

$$\begin{gathered}\frac{\partial \theta}{\partial \tau} + Y \frac{\partial \theta}{\partial \xi} + 2Y \frac{\partial}{\partial \xi}\int_0^1 [\Phi(\eta) - 1]\,(c - \theta)\,\eta\, d\eta = 0\\ \theta = 2\int_0^1 c\eta\, d\eta, \quad \theta = \theta(\tau, \xi)\end{gathered} \tag{1.5}$$

Here θ is the concentration of admixture, average across the section of the pipe, which depends only on the variables τ and ξ. Combining the terms of Eqs. (1.4) and (1.5), we obtain the following equation for the function $\Psi = c - \theta$:

$$\begin{gathered}\frac{\partial \Psi}{\partial \tau} - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta D_* \frac{\partial \Psi}{\partial \eta}\right) = 2Y \frac{\partial}{\partial \xi}\int_0^1 [\Phi(\eta) - 1]\,\Psi\eta\, d\eta\\ - Y\,[\Phi(\eta) - 1]\,\frac{\partial c}{\partial \xi} - Y \frac{\partial \Psi}{\partial \xi}\end{gathered} \tag{1.6}$$

Equation (1.6) is considered as a nonhomogeneous equation with a free term on the right side, and is solved by the method of successive approximations.

2. Taylor took the averaged concentration c, equal to the mean concentration θ, as the first approximation when determining the concentration profile. Justification for such an approximation is the fact that for instants of time which are much larger than the diffusion constant a^2/D_0, because of the radial diffusion, inhomogeneities of the concentration in the cross section of the pipe are nearly levelled out, and in the flow there are only insignificant deviations of the concentration from the average value in the cross section of the tube. These deviations are caused by the inhomogeneous convective transfer of the admixture due to the difference of the averaged and mean velocities.

We use this approximation to find the subsequent solutions. With this aim we substitute $c = \theta$ in the right side of Eq. (1.6). As a result of the substitution, we obtain the following equation:

$$\frac{\partial \Psi}{\partial \tau} - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta D_* \frac{\partial \Psi}{\partial \eta}\right) = -Y\,[\Phi(\eta) - 1]\,\frac{\partial \theta}{\partial \xi} \tag{2.1}$$

We seek the solution of this equation which satisfies the limiting conditions

$$\Psi\left.\right|_{\tau=0} = 0, \quad \left.\frac{\partial \Psi}{\partial \eta}\right|_{\eta=0} = \left.\frac{\partial \Psi}{\partial \eta}\right|_{\eta=1} = 0 \tag{2.2}$$

When solving the problem (2.1), (2.2), in contrast to the Taylor solution, we retain the local time-derivative of the function in the right side of Eq. (2.1). The sought solution has the form

$$\Psi(\tau, \xi, z) = -Y \sum_{n=1}^{\infty} \frac{a_n X_n(z)}{\alpha_n^2 \| X_n \|^2} \left[\frac{\partial\theta}{\partial\xi} - \frac{\partial}{\partial\tau} \int_0 \exp[-\alpha_n^2(\tau - s)] \frac{\partial\theta}{\partial\xi} ds \right] \tag{2.3}$$

$$a_n = \int_0^{1/4} X_n(z)[\Phi(z) - 1]\, dz, \quad z = \frac{1}{4}\eta^2$$

Here X_n and α_n are the eigenfunctions and eigenvalues of the following Sturm–Liouville problem:

$$\frac{d}{dz}\left[zD_* \frac{dX}{dz}\right] + \alpha^2 X = 0, \quad X'(0) = X'\left(\frac{1}{4}\right) = 0 \tag{2.4}$$

The transfer coefficient D_* is assumed to depend only on the radial coordinate. With the aid of the solution (2.3) we can approximately find the concentration profile in terms of the unknown function $\theta(\tau, \xi)$. To determine the mean concentration θ in the section of the pipe, the solution (2.3) is substituted into the integral of the third term of Eq. (1.5).

As a result, we obtain an equation for determining the concentration $\theta(\tau, \xi)$. This equation has the form

$$\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} = 4Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\alpha_n^2 \| X_n \|^2} \left[\frac{\partial^2\theta}{\partial\xi^2} - \frac{\partial}{\partial\tau} \int_0^{\tau} \exp[-\alpha_n^2(\tau - s)] \frac{\partial^2\theta(s, \xi)}{\partial\xi^2} ds \right] \tag{2.5}$$

It is not difficult to show that the solutions of this equation for $\tau \to \infty$ are asymptotically close to a diffusion equation of the following form:

$$\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} = 4Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\alpha_n^2 \| X_n \|^2} \frac{\partial^2\theta}{\partial\xi^2} \tag{2.6}$$

3. We calculate quantity a_n which enters the expression for the diffusion coefficient. With this aim we substitute the function X_n determined from Eq. (2.4) into the integrand of (2.3). We have

$$a_n = -\frac{1}{2\alpha_n^2} \int_0^1 \frac{d}{d\eta}\left[\eta D_* \frac{dX_n}{d\eta}\right][\Phi(\eta) - 1]\, d\eta \tag{3.1}$$

Integrating by parts, we obtain

$$a_n = \frac{1}{2\alpha_n^2} \int_0^1 \eta D_* X_n'(\eta)\Phi'(\eta)\, d\eta \tag{3.2}$$

The coefficient D_* equals the sum of the molecular and turbulent diffusion coefficients

$$D_* = D / D_0 + D_t / D_0 \tag{3.3}$$

In a turbulent flow the transfer of matter, heat, and impulse takes place almost with equal intensity. Therefore, according to the well-known Reynolds analogy, the coefficients characterizing the intensity of transfer of these quantities are taken as equal.

On the basis of this analogy, using the Boussinesq expression, we write the coefficient of radial turbulent diffusion in the form

$$D_t = -\frac{au_*^2\eta}{U}[\Phi'(\eta)]^{-1} \tag{3.4}$$

With the expressions (3.1), (3.4) taken into account, we write the expression for a_n in the following form:

$$a_n = \frac{1}{2\alpha_n^2}\left[\frac{D}{D_0}\int_0^1 \eta X_n'(\eta)\Phi'(\eta)\, d\eta - \frac{u_*^2 a}{UD_0} X_n(1)\right]$$

4. The unknown solution of the Sturm–Liouville problem, $X_n(\eta)$, enters the expression for calculating the quantity a_n and the other expressions derived above. The difficulty of solving the problem (2.4) is

caused by the fact that the radial coefficient of turbulent diffusion depends on the variable z. The experiments by Laufer and Nunner [12], as well as the investigations carried out in [13,14], allow us to establish the character of this dependence. The coefficient of radial diffusion varies in the most substantial manner close to the inner surface of the tube, in a region which is the narrower, the larger is the Reynolds number, while in the core of the turbulent flow occupying an overwhelming proportion of the pipe cross section the coefficient of turbulent diffusion is almost constant. Such a character of dependence of D_* on z provides justification, when solving the problem (2.4), for taking the coefficient of radial turbulent diffusion in the whole section of the pipe as constant and equal to its mean value in the section of the pipe. It is natural to take the typical value of the diffusion coefficient as its mean value in the section of the pipe, i.e., to take D_* equal to unity. For a constant value of D_* we easily find the eigenfunctions and eigenvalues of the Sturm–Liouville problem (2.4). We have

$$X_n(z) = J_0(2\alpha_n z^{1/2}), \quad J_1(\alpha_n) = 0 \tag{4.1}$$

Here $J_\nu(z)$, $\nu = 0, 1$ are Bessel functions of the first kind. The square of the norm of the eigenfunctions thus found equals

$$\|X_n\|^2 = {}^1/_4 J_0^2(\alpha_n)$$

We calculate the effective diffusion coefficient. We have

$$\frac{K}{D_0} = 16Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\alpha_n^2 J_0^2(\alpha_n)} = 4Y^2 \frac{u_*^4 a^2}{U^2 D_0^2} \times \sum_{n=1}^{\infty} \frac{1}{\alpha_n^6} \left[1 + \frac{DU}{u_*^2 a J_0(\alpha_n)} \int_0^1 \eta J_1(\alpha_n \eta) \Phi'(\eta)\, d\eta\right]^2 \tag{4.2}$$

In this expression the second term inside the square brackets is due to molecular diffusion. The first eigenvalue is $\alpha_1 = 3.83$, the values of the subsequent roots are 7.02, 10, 17, etc. The eigenvalues increase so rapidly that the computation of the series (4.2) can be confined to the first term. We have

$$\frac{K}{D_0} \approx 4Y^2 \frac{u_*^4 a^2}{U^2 D_0^2 \alpha_1^6} \left[1 + \frac{DU}{u_*^2 a J_0(\alpha_1)} \int_0^1 \eta J_1(\alpha_1 \eta) \Phi'(\eta)\, d\eta\right]^2 \tag{4.3}$$

To calculate the integral inside the square brackets, we must choose a profile of the averaged velocity of the turbulent flow. We use, for example, the universal law of velocity distribution for which the function $\Phi(\eta)$ has the form

$$\Phi(\eta) = U_0 / U + (2.5u_* / U) \times \ln(1 - \eta)$$

Here U_0 is the velocity of flow on the axis of the pipe.

We substitute the derivative of this function into the integrand and carry out integration by a numerical method. We obtain

$$\int_0^1 \eta J_1(3.83\eta) \Phi'(\eta)\, d\eta \approx -1.65 u_* / U \tag{4.4}$$

Taking into account this result, and also the fact that $J_0(3.83) \approx -0.401$, we rewrite the expression (4.3) as follows:

$$\frac{K}{D_0} \approx 4Y^2 \frac{u_*^4 a^2}{(3.83)^6 U^2 D_0^2} \left(1 + 4.12 \frac{D}{u_* a}\right)^2 \tag{4.5}$$

To calculate the effective diffusion coefficient according to the expression (4.5), we must find the value of the coefficient D_0. To determine the value of D_0 we use the profile of the coefficient of radial turbulent diffusion from Taylor's paper. In this case

$$D_0 = 0.052\, u_* a + D$$

Substituting the value of D_0 into the expression (4.5), we obtain the expression for the dimensionless value of the effective diffusion coefficient

$$\frac{K}{2aU}=5.2\sqrt{\frac{\lambda}{8}}\left[1-\frac{30.4}{\mathrm{Sc}\,\mathrm{Re}}\left(\frac{\lambda}{8}\right)^{-1/2}\right]^2, \quad \mathrm{Sc}=\frac{\nu}{D} \tag{4.6}$$

Here λ is the hydraulic resistance. This expression differs from the well-known Taylor expression by the additional term which takes into account the dependence of the effective coefficient on the Schmidt number.

The method used to derive the expression (4.6), in contrast to the method used in Taylor's work, is not connected with the choice of this or that law of distribution of the averaged velocity. This circumstance allows us to generalize the relationship thus found for the region of large Reynolds numbers for which the velocity distribution and hydraulic resistance essentially depends on the roughness. With this aim, just as it was done in [15], we represent the coefficient of radial turbulent diffusion in the form of a sum of the coefficient of turbulent diffusion in a smooth pipe and the coefficient of turbulent diffusion depending on the roughness. The latter coefficient has the form

$$D_S=0.39u_*a\left[\varepsilon-\frac{7.8}{\mathrm{Re}}\left(\frac{\lambda}{8}\right)^{-1/2}\right](0.03+0.97\eta) \qquad \varepsilon=\frac{\Delta}{a} \tag{4.7}$$

Here ε is the relative roughness.

The average value of this coefficient for a cross section of the pipe is

$$D_{0S}\approx 0.265u_*a\left[\varepsilon-\frac{7.8}{\mathrm{Re}}\left(\frac{\lambda}{8}\right)^{-1/2}\right] \tag{4.8}$$

Taking into account this equation, we write the coefficient D_0 for flow in a rough pipe as follows:

$$D_0=\left[0.052+0.265\left(\varepsilon-\frac{7.8}{\mathrm{Re}}\left(\frac{\lambda}{8}\right)^{-1/2}\right)\right]u_*a \tag{4.9}$$

This expression is substituted into (4.5). As a result, we obtain the equation for the effective diffusion coefficient which generalizes the Taylor expression for the case of flow in rough pipes

$$K_*=\frac{K}{2aU}=5.2\sqrt{\frac{\lambda}{8}}\left[1+5.1\left(\varepsilon-\frac{7.8}{\mathrm{Re}}\left(\frac{\lambda}{8}\right)^{-1/2}\right)\right]^{-3} \tag{4.10}$$

The value of the hydraulic resistance for various Reynolds numbers and parameters of relative roughness is represented in the form of tables in [15]. Numerous expressions for the hydraulic resistance of different tubes are presented in the book [16].

5. Taking into account the expression derived above for the effective coefficient of turbulent diffusion (4.5) and also the rapid growth in the eigenvalues α_n, we write Eq. (2.5) in the following form (the molecular transfer is not taken into account)

$$\frac{\partial\theta}{\partial\tau}+Y\frac{\partial\theta}{\partial\xi}=\varkappa^2\int_0^\tau \exp\left[-\alpha_1{}^2(\tau-s)\right]\frac{\partial^2\theta}{\partial\xi^2}ds$$
$$\varkappa^2=\frac{4u_*{}^4a^2}{\alpha_1{}^4U^2D_0{}^2}Y^2 \tag{5.1}$$

In a coordinate system moving with the mean velocity of the flow, Eq. (5.1) has the form

$$\frac{\partial\theta}{\partial\tau}=\varkappa^2\int_0^\tau \exp\left[-\alpha_1{}^2(\tau-s)\right]\frac{\partial^2\theta}{\partial z^2}ds, \quad z=\xi-Y\tau \tag{5.2}$$

Here z is the distance in the moving coordinate system. We differentiate with respect to τ the terms on the left and right sides of Eq. (5.2). As a result of some simple transformations we obtain the following equation:

$$\frac{\partial^2\theta}{\partial\tau^2}+\alpha_1{}^2\frac{\partial\theta}{\partial\tau}=\varkappa^2\frac{\partial^2\theta}{\partial z^2} \tag{5.3}$$

This is the so-called telegraph equation which in the given case describes the distribution of the mean concentration in the section of the pipe. In contrast to the diffusion equation of Taylor, Eq. (5.3) is of the hyperbolic type.

From the solution of this equation we obtain the final velocity of propagation of the admixture toward both sides from the zero section. This velocity equals $\varkappa$.

It can be shown that the continuous part of the solution of Eq. (5.3) differs from the solution of Taylor's equation at the beginning of the mixing process, when $\tau < 1$. This difference is the more substantial, the smaller is τ. For a mixing process in main pipe lines, where concentration distribution for $\tau \gg 1$ is of practical interest, this difference between the distribution in the initial stage of the process is not important. Therefore we can use Taylor's model, making use of the expression (4.10) derived here for the effective coefficient. However, when investigating the process of mixing in petrochemical reactors of a tubular type, where the time of the mixing process, τ, is comparable with unity, it is advisable to use the solution of the dispersion equation (5.3). The form of the solution of Eq. (5.3) depends on the form of the signal at the input of the reactor. However, in any particular case the finding of the solutions presents no special difficulties.

LITERATURE CITED

1. H. Kramers and K. Vestertern, Chemical Reactors, Their Calculation and Control [Russian translation], Khimiya, Moscow (1967).
2. G. Taylor, "The dispersion of matter in turbulent flow through a pipe," Proc. Roy. Soc., Ser. A, 223, No. 1155 (1954).
3. L. E. Flint and P. Eisenklam, "Dispersion of matter in transitional flow through straight tubes," Proc. Roy. Soc., Ser. A, 315, No. 1523, 519-533 (1970).
4. V. A. Yufin, "Mixing during successive pump-over of oil and oil products in pipe lines," Neft. Khoz-vo, No. 8 (1957).
5. A. K. Gallyamov and V. I. Maron, "On the value of the effective coefficient of turbulent diffusion," Neft. i Gaz, No. 8 (1968).
6. K. D. Frolov, "Mixing of oil products with various viscosity in pipes," Transport i Khranenie Nefti i Nefteproduktov, No. 12 (1964).
7. O. Levenspiel, "Longitudinal mixing of fluids flowing in circular pipes," Industr. Engng. Chem., 50, No. 3 (1958).
8. V. S. Yablonskii, A. Sh. Asaturyan, and Kh. O. Khizgilov, "On turbulent diffusion in pipes," Inzh.-Fiz. Zh., No. 3 (1960).
9. V. W. Goldschmidt and M. K. Householder, "Longitudinal dispersion for turbulent flow in pipes," Industr. Engng. Chem. Fundam., 8, No. 1 (1969).
10. V. Ananthakrishnan, W. N. Gill, and A. J. Barduhn, "Laminar dispersion in capillaries," Part 1, A.I.Ch.E. J., 11, No. 6 (1965).
11. L. J. Tichacek, C. H. Barkelew, and T. Baron, "Axial mixing in pipes," A.I.Ch.E. J., 3, No. 4 (1957).
12. J. O. Hinze, Turbulence, McGraw-Hill, New York (1959).
13. W. G. Schlinger and B. H. Sage, "Material transfer in turbulent gas streams. Concentric flow," Industr. Engng. Chem., 45, No. 3 (1953).
14. S. Lynn, W. H. Corcoran, and B. H. Sage, "Material transport in turbulent gas streams: radial diffusion in a circular conduit," A. I. Ch. E. J., 3, No. 1 (1957).
15. M. D. Millionshchikov, Turbulent Flows in a Boundary Layer and in Pipes [in Russian], Nauka, Moscow (1969).
16. A. D. Al'tshul, Hydraulic Resistance [in Russian], Nedra, Moscow (1970).